Computer Science 294 Lecture 22 Notes

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1 Hardness of Approximation for Max-Cut and The Majority is Stablest Theorem

1.1 Proof sketch of the invariance principle

Let's finish our proof sketch of the invariance principle from last time.

Theorem 1.1 (Invariance principle). *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a multilinear polynomial of degree d, i.e.*

$$f(x) = \sum_{S \subseteq [n]} \widehat{f}(S) \prod_{i \in S} x_i.$$

Let $X_1, \dots, X_n \sim \{\pm 1\}$ be independent random bits, and let $Y_1, \dots, Y_n \sim N(0,1)$ be independent standard Gaussians. Then

$$|\mathbb{E}[\psi(f(X_1, \dots, X_n))] - \mathbb{E}[\psi(f(Y_1, \dots, Y_n))]| \leq \frac{\|\psi^{(4)}\|_\infty}{24} \cdot 9^{d-1} \cdot \sum_{i=1}^n \mathrm{Inf}_i^2(f)(\mathbb{E}[X_i^4] + \mathbb{E}[Y_i^4]),$$

where $\mathrm{Inf}_i(f) = \sum_{S \ni i} \widehat{f}(S)^2$.

Proof sketch of invariance principle. We want to show

$$\mathbb{E}_{X_1, \dots, X_n \sim \{\pm 1\}}[\psi(f(X_1, \dots, X_n))] \approx \mathbb{E}_{Y_1, \dots, Y_n \sim N(0,1)}[\psi(f(Y_1, \dots, Y_n))],$$

so define the hybrids

$$H_i = f(Y_1, \dots, Y_i, X_{i+1}, \dots, X_n).$$

As before, it suffices to show that for all i, $\mathbb{E}[\psi(H_{i-1})] \approx \mathbb{E}[\psi(H_i)]$. We can write

$$f(x) = x_i D_i f(x) + E_i f(x),$$

where $D_i f(X)$ and $E_i f(X)$ don't depend on X_i. Since H_i and H_{i-1} only differ in the i-th coordinate, we have

$$H_i = Y_i D_i f(Y_1, \dots, Y_{i-1} X_{i+1}, \dots, X_n) + E_i f(Y_1, \dots, Y_{i-1}, X_{i+1}, \dots, X_n),$$

$$H_{i-1} = X_i D_i f(Y_1, \dots, Y_{i-1} X_{i+1}, \dots, X_n) + E_i f(Y_1, \dots, Y_{i-1}, X_{i+1}, \dots, X_n),$$

Now write

$$H_i = Y_i \cdot \Delta + U, \qquad H_{i-1} = X_i \cdot \Delta + U,$$

where

$$U = E_i f(Y_1, \dots, Y_{i-1}, X_{i+1}, \dots, X_n), \qquad \Delta = D_i f(Y, \dots, Y_{i-1}, X_{i+1}, \dots, X_n).$$

Now

$$||E[\psi(H_{i-1})] - \mathbb{E}[\psi(H_i)]| = |\,\mathbb{E}[\psi(U + X_i\Delta)] - \mathbb{E}[\psi(U + Y_i\Delta)]|$$

Using the Taylor expansion of ψ around U,

$$\begin{aligned} &\le \frac{\|\psi^{(4)}\|_\infty}{4!}(\mathbb{E}[(X_i\Delta)^4] + \mathbb{E}[(Y_i\Delta)^4]) \\ &\le \frac{\|\psi^{(4)}\|_\infty}{4!}(\mathbb{E}[X_i^4]\,\mathbb{E}[\Delta^4] + \mathbb{E}[Y_i^4]\,\mathbb{E}[\Delta^4]) \end{aligned}$$

By Bonami's lemma, $\mathbb{E}[\Delta^4] \le 9^{d-1}\,\mathbb{E}[\Delta^2]^2$. By Parseval's identity, $\mathbb{E}[\Delta^2] = \sum_{S \ni i} \widehat{f}(S)^2 = \mathrm{Inf}_i(f)$.

$$\begin{aligned} &\le \frac{\|\psi^{(4)}\|_\infty}{4!}(9^{d-1}(\mathrm{Inf}_i(f))^2 + 3 \cdot 9^{d-1}(\mathrm{Inf}_i(f))^2) \\ &= \|\psi^{(4)}\|_\infty \frac{4}{4!}(9^{d-1}\,\mathrm{Inf}_i(f))^2. \end{aligned}$$

□

1.2 Hardness of approximation for Max-Cut

The Max-Cut problem is that given a graph, we want to label the vertices with $+1$ or -1 so that the number of edges between $+1$ and -1 vertices is maximized. To show that Max-Cut is hard to approximate, it suffices to design a dictator-vs-no-notable-coordinates test using "$\neq$" predicates such that

1. If f is a Dictator, then

$$\mathbb{P}(\text{tester accepts } f) \ge \frac{1}{2} + \frac{1}{2}\rho,$$

2. If f has no ε-notable coordinates (i.e. $\mathrm{Inf}_i^{(1-\varepsilon)}(f) \le \varepsilon$ for all i), then

$$\mathbb{P}(\text{tester accepts } f) \le 1 - \frac{\arccos(\rho)}{\pi} + \lambda(\varepsilon),$$

   where $\lambda(\varepsilon) \to 0$ as $\varepsilon \to 0$.

The test is as follows:

1. Pick a noise parameter $-1 < \rho' \leq 0$ (think $\rho' = -\rho$).
2. Pick a ρ'-correlated pair $X, Y \sim \{\pm 1\}^n$.
3. Accept if and only if $f(X) \neq f(Y)$.

With this test,

$$\begin{aligned}\mathbb{P}(\text{tester accepts } f) &= \mathbb{E}\left[\frac{1}{2} - \frac{1}{2} f(X) f(Y)\right] \\ &= \frac{1}{2} - \frac{1}{2}\operatorname{Stab}_{\rho'}(f).\end{aligned}$$

Now we analyze by cases:

1. If f is a dictator, then

$$\mathbb{P}(\text{tester accepts } f) = \frac{1}{2} - \frac{1}{2}\operatorname{Stab}_{\rho'}(f) = \frac{1}{2} - \frac{1}{2}\rho'.$$

2. If f has no ε-notable coordinates, we want to show that

$$\frac{1}{2} - \frac{1}{2}\operatorname{Stab}_{\rho'}(f) \leq 1 - \frac{1}{\pi}\arccos(\rho) + \lambda(\varepsilon).$$

   Rearranging this, we want to show that

$$-\operatorname{Stab}_{\rho'}(f) \leq 1 - \frac{2}{\pi}\arccos(\rho) + 2\lambda(\varepsilon).$$

   The Fourier expansion of the negative stability is

$$\operatorname{Stab}_{\rho'}(f) = -\underbrace{W^0}_{\leq 0} - \underbrace{\rho' W^1(f)}_{\geq 0} - \underbrace{(\rho')^2 W^2(f)}_{\leq 0} + \cdots$$

   Dropping the negative terms, it suffices to prove that

$$\rho W^1(f) + \rho^3 W^3(f) + \rho^5 W^5(f) + \cdots \leq 1 - \frac{2}{\pi}\arccos(\rho) + 2\lambda(\varepsilon).$$

   This looks like the ρ-stability of f when we only take the odd part of f. Note that f^{odd} is bounded because $f_{\text{odd}} = \frac{f(x)-f(-x)}{2}$, which is bounded for a boolean function f.

1.3 Majority is stablest

We will now prove the following theorem.

Theorem 1.2 (Majority is stablest, MOO)**.** *Let $f : \{\pm 1\}^n \to [-1,1]$ be such that f is odd and $\mathrm{Inf}_i^{(1-\varepsilon)}(f) \le \varepsilon$ for all i. Then, for all $0 \le \rho \le 1$,*

$$\mathrm{Stab}_\rho(f) \le \underbrace{1 - \frac{2}{\pi}\arccos(\rho)}_{=\lim_{n\to\infty}\mathrm{Stab}_\rho(\mathrm{MAJ}_n)} + 2\lambda(\varepsilon),$$

where $\lambda(\varepsilon) \to 0$ as $\varepsilon \to 0$.

What is the analogous statement on Gaussian space?

Theorem 1.3 (Borrell '85)**.** *Let $f : \mathbb{R}^n \to [-1,1]$ with $\mathbb{E}[f(Z)] = 0$, where Z is an n-dimensional Gaussian distribution. Then for all $\rho \ge 0$,*

$$\mathrm{GStab}_\rho(f) := \mathbb{E}[f(Z)f(Z')] \le 1 - \frac{2}{\pi}\arccos(\rho),$$

where (Z_i, Z_i') are ρ-correlated Gaussians (independent of the other coordinates).

Recall Sheppard's theorem, which tells us that if $f(x) = \mathrm{sgn}(x_1 + \cdots + x_n)$, then

$$\mathrm{GStab}_\rho(f) = 1 - \frac{2}{\pi}\arccos(\rho).$$

Proof of Majority is Stablest via Borrell's theorem. Given $f : \{\pm 1\}^n \to [-1,1]$ where f is odd and $\mathbb{E}[f] = 0$, think of f as a multilinear polynomial

$$f(x) = \sum_{S \subseteq [n]} \widehat{f}(S) \prod_{i \in S} x_i.$$

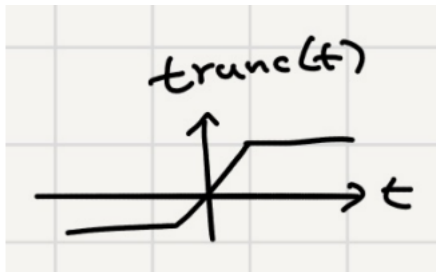
We assumed that $\mathrm{Inf}_i^{(1-\varepsilon)}(f) \le \varepsilon$ for all i. Using the polynomial interpretation of f,

$$\begin{aligned}
\mathrm{GStab}_\rho(f) &= \mathbb{E}_{(Z,Z')\ \rho\text{-corr.}}[f(Z)f(Z')] \\
&= \sum_S \widehat{f}(S)^2 \rho^{|S|} \\
&= \mathrm{Stab}_\rho(f).
\end{aligned}$$

To use Borrell's theorem, we need to know that $f : \mathbb{R}^n \to [-1,1]$. On the Boolean domain, we need that for all $x \in \{\pm 1\}$ that $f(x) \in [-1,1]$. Thus, we can hope that with high probability $f(Z) \in [-1,1]$ for $Z = (Z_1, \ldots, Z_n)$ Gaussians.

Let $\overline{f}(z) = \mathrm{trunc}(f(z))$ be the truncated function, where

$$\mathrm{trunc}(t) = \begin{cases} -1 & t \leq -1 \\ t & -1 < t < 1 \\ 1 & t \geq 1. \end{cases}$$



By Borell's theorem,

$$\mathrm{GStab}_\rho(\overline{f}) \leq 1 - \frac{2}{\pi}\arccos(\rho),$$

so it suffices to show that $\mathrm{GStab}_\rho(\overline{f})$ is similar to $\mathrm{GStab}_\rho(f)$. We'll show that

$$\mathbb{E}[(f(Z) - \overline{f}(Z))^2] \leq o_\varepsilon(1),$$

which gives

$$\begin{aligned} |\,\mathrm{GStab}_\rho(f) - \mathrm{GStab}_\rho(\overline{f})| &= |\,\mathbb{E}[f(Z)f(Z') - \overline{f}(Z)\overline{f}(Z')]| \\ &\leq |\,\mathbb{E}[f(Z)f(Z') - f(Z)\overline{f}(Z')]| + |\,\mathbb{E}[f(Z)\overline{f}(Z') - \overline{f}(Z)\overline{f}(Z')]| \\ &\leq |\,\mathbb{E}[f(Z)(f(Z') - \overline{f}(Z'))]| + |\,\mathbb{E}[(f(Z) - \overline{f}(Z))\overline{f}(Z')]| \end{aligned}$$

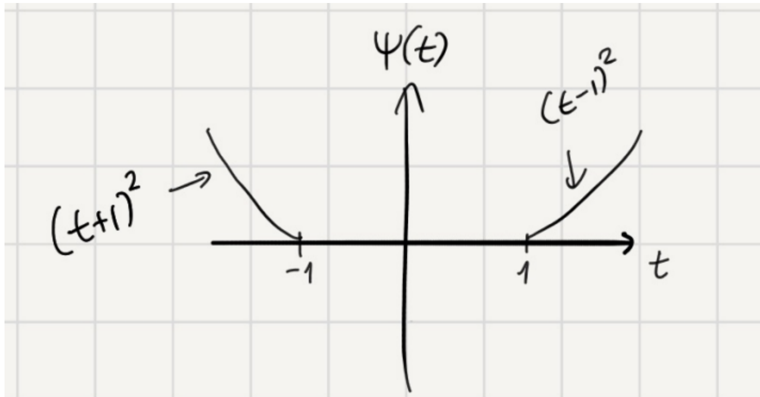
By Cauchy-Schwarz,

$$\begin{aligned} &\leq \sqrt{\mathbb{E}[f(Z)^2]}\sqrt{\mathbb{E}[(f(Z') - \overline{f}(Z'))^2} + \sqrt{\mathbb{E}[\overline{f}(Z)^2]}\sqrt{\mathbb{E}[(f(Z) - \overline{f}(Z))^2} \\ &= \sqrt{\sum_S \widehat{f}(S)^2}\sqrt{o_\varepsilon(1)} + \sqrt{\sum_S \widehat{f}(S)^2}\sqrt{o_\varepsilon(1)} \\ &= \sqrt{E[f(X)^2]}\sqrt{o_\varepsilon(1)} + \sqrt{E[f(X)^2]}\sqrt{o_\varepsilon(1)} \\ &\leq \sqrt{o_\varepsilon(1)} \end{aligned}$$

To prove the claim, define

$$\psi(t) = \begin{cases} (t+1)^2 & t < -1 \\ 0 & -1 \leq t \leq 1 \\ (t-1)^2 & t > 1. \end{cases}$$

$$= \text{dist}(t, [-1,1])^2$$



Then

$$\mathbb{E}[\psi(f(Z))] = \mathbb{E}[(f(Z) - \overline{f}(Z))^2].$$

We know that $\mathbb{E}[\psi(f(X))] = 0$. Can we get by the invariance principle that $\mathbb{E}[\psi(f(Z))] \leq o_\varepsilon(1)$? This test function is not smooth enough, but we can slightly alter it. The idea is to apply some smal noise $\delta = \delta(\varepsilon)$, where $\delta \gg \varepsilon$ but $\delta \to 0$ as $\varepsilon \to 0$ (e.g. $\delta = \frac{1}{\log\log(1/\varepsilon)}$). Set $g = T_{1-\delta} f$. From the assumption $\text{Inf}_i^{(1-\varepsilon)}(f) \leq \varepsilon$, we get

$$\begin{aligned}\text{Inf}_i(g) &= \sum_{S \ni i} \widehat{f}(S)^2 \\ &= \sum_{S \ni i} (1-\delta)^{2|S|} \widehat{f}(S)^2\end{aligned}$$

Since $\delta \ll \varepsilon$,

$$\begin{aligned}&\leq \sum_{S \ni i} (1-\varepsilon)^{|S|} \widehat{f}(S)^2 \\ &\leq \sum_{S \ni i} (1-\varepsilon)^{|S|-1} \widehat{f}(S)^2 \\ &\leq \varepsilon\end{aligned}$$

We want to show that $\text{Stab}_\rho(f) \approx \text{Stab}_\rho(g)$. We have

$$\begin{aligned}|\text{Stab}_\rho(f) - \text{Stab}_\rho(g)| &= \left| \sum_S \widehat{f}(S)^2 \rho^{|S|} (1 - (1-\delta)^{2|S|}) \right| \\ &\leq \max_{k \in \{0,\dots,n\}} \rho^k (1 - (1-\delta)^{2k}) \\ &\leq \max_k \rho^k \cdot 2k \cdot \delta\end{aligned}$$

$$\le O(\delta)$$

The next step is to turn g into a low degree polynomial by removing its high degree parts. Set $h := g^{\le 1/\delta^2}(x)$. Then h is odd, so $\mathbb{E}[h] = 0$. Overall, we have

$$\mathrm{Stab}_\rho(f) \approx \mathrm{Stab}_\rho(h) = \mathrm{GStab}_\rho(h) \approx \mathrm{GStab}_\rho(\overline{h}) \le 1 - \frac{2}{\pi}\arccos(\rho).$$

The step $\mathrm{GStab}_\rho(h) \approx \mathrm{GStab}_\rho(\overline{h})$ comes from

$$\begin{aligned}
\mathbb{E}_{Z\sim N(0,1)^n}[\psi(h(Z))] &\approx \mathbb{E}_{X\sim\{\pm1\}^n}[\psi(h(X))] \\
&= \mathbb{E}_X[(h(X) - \overline{h}(X))^2] \\
&\le \mathbb{E}_X[(h(X) - g(X))^2] \\
&\le \sum_{|S|>1/\delta^2} \widehat{f}(S)^2 \cdot (1-\delta)^{2|S|} \\
&\le (1-\delta)^{1/\delta^2} \\
&\le \delta.
\end{aligned}$$

The error in the invariance principle is $\le \frac{\|\psi\|_\infty}{24} 9^{1/\delta^2} \cdot 4 \cdot \sum_{i=1}^n \mathrm{Inf}_i(h)^2$. Using the fact that $\mathrm{Inf}_i(h) \le \varepsilon$ and $\varepsilon \ll \delta$, this is $O(9^{1/\delta^2} \cdot 1/\delta^2 \cdot \varepsilon)$, which is $\le \varepsilon^{0.99}$. $\square$

We should think of this as the kind of proof which has a central idea and proceeds by trial and error.